

Cancellation of projective modules over non-Noetherian rings

Manoj K. Keshari

Department of Mathematics, IIT Bombay, Mumbai - 400076, India; keshari@math.iitb.ac.in

Abstract: Let R be a ring of dimension d and A be one of $R[Y]$ or $R[Y, Y^{-1}]$. If P is a projective A -module of rank $\geq d + 1$, then we show that $E(A \oplus P)$ acts transitively on $\text{Um}(A \oplus P)$ (see 4.8). When P is free, this result is due to Yengui [22] (when $A = R[Y]$) and Abedelfatah [1] (when $A = R[Y, Y^{-1}]$).

1 Introduction

All rings are assumed to be commutative with unity and all modules are finitely generated. The dimension of a ring will mean its Krull dimension and all projective modules will be of constant rank.

Let R be a Noetherian ring of dimension d , $A = R[Y_1, \dots, Y_r, (f_1 \cdots f_s)^{-1}]$ with $s \leq r$, $f_i \in R[Y_i]$ and P a projective A -module of rank $\geq \max\{2, d + 1\}$. Then author-Dhorajia [5] proved that $E(A \oplus P)$ acts transitively on $\text{Um}(A \oplus P)$. In particular, P is cancellative, i.e. $P \oplus A^n \xrightarrow{\sim} Q \oplus A^n$ for some A -module $Q \implies P \xrightarrow{\sim} Q$. The case $r = s = 0$ is due to Bass [4], the case $r = 1, s = 0$ is due to Plumstead [14], the case $r = s = 1, f_1 = Y_1$ is due to Mandal [13] (he proved that P is cancellative), the case $s = 0$ is due to Rao [16] (he proved that P is cancellative), the Laurent polynomial case $f_i = Y_i$ is due to Lindel [12].

Heitmann [8] generalized Bass' result to all commutative rings without Noetherian condition. It is natural to ask if analog of above results hold for non-Noetherian rings. There are some results in this direction.

Let R be a ring of dimension 0 and $A = R[Y_1, \dots, Y_n]$. Then Ellouz-Lombardi-Yengui [7] proved that all projective A -modules are free, generalizing the well known Quillen-Suslin result [15, 19]. Abedelfatah [2] generalized this by proving that $E_r(A)$ acts transitively on $\text{Um}_r(A)$ for $r \geq 3$. We will further generalize this result and prove the following (3.2, 3.3):

Let R be a ring of dimension 0 and $A = R[Y_1, \dots, Y_n, (f_1 \cdots f_m)^{-1}]$, where $m \leq n$ and $f_i \in R[Y_i]$. Then all projective A -modules are free and $E_r(A)$ acts transitively on $\text{Um}_r(A)$ for $r \geq 3$.

Let R be a ring of dimension d and $n \geq d + 2$. Then Yengui [22] proved that $E_n(R[X])$ acts transitively on $\text{Um}_n(R[X])$ and Abedelfatah [1] proved that $E_n(R[X, X^{-1}])$ acts transitively on $\text{Um}_n(R[X, X^{-1}])$. We will generalize both results as follows (4.8) which are analog of Plumstead [14] and Mandal [13] results (mentioned above) in non-Noetherian case.

Let R be a ring of dimension d and A be one of $R[X]$ or $R[X, X^{-1}]$. If P is a projective A -module of rank $\geq d + 1$, then $E(A \oplus P)$ acts transitively on $\text{Um}(A \oplus P)$. In particular, P is cancellative.

For a Prufer domain R of dimension d , we will further generalize above result as follows (4.9): *Let $f \in R[X]$ and $A = R[X, f^{-1}]$. If P is a projective A -module of rank $\geq d + 1$, then $E(A \oplus P)$ acts transitively on $\text{Um}(A \oplus P)$.*

2 Preliminaries

For readers convenience, we begin with some definitions. Let A be a ring and M an A -module. We say that $m \in M$ is *unimodular* if there exists $\phi \in M^* = \text{Hom}_A(M, A)$ such that $\phi(m) = 1$. The set of all unimodular elements of M will be denoted by $\text{Um}(M)$. For an ideal J of A , we denote by $\text{Um}^1(A \oplus M, J)$, the set of all $(a, m) \in \text{Um}(A \oplus M)$ such that $a \in 1 + J$ and by $\text{Um}(A \oplus M, J)$, the set of all $(a, m) \in \text{Um}^1(A \oplus M)$ with $m \in JM$. We will write $\text{Um}_r(A, J)$ for $\text{Um}(A \oplus A^{r-1}, J)$.

We denote by $\text{Aut}_A(M)$, the group of all A -automorphism of M . For an ideal J of A , we denote by $E^1(A \oplus M, J)$, the subgroup of $\text{Aut}_A(A \oplus M)$ generated by all the automorphisms $\Delta_{a\varphi}$ and Γ_m , where

$$\Delta_{a\varphi} = \begin{pmatrix} 1 & a\varphi \\ 0 & id_M \end{pmatrix} \quad \text{and} \quad \Gamma_m = \begin{pmatrix} 1 & 0 \\ m & id_M \end{pmatrix} \quad \text{with } a \in J, \varphi \in M^*, m \in M.$$

In particular, let $E_{r+1}(A)$ denote the subgroup of $\text{SL}_{r+1}(A)$ generated by elementary matrices $I + ae_{ij}$, where $a \in A$ and e_{ij} for $i \neq j$ is the matrix with only non-zero entry 1 at (i, j) -th place. Then we denote by $E_{r+1}^1(A, J)$, the subgroup of $E_{r+1}(A)$ generated by $\Delta_{\mathbf{a}}$ and $\Gamma_{\mathbf{b}}$, where

$$\Delta_{\mathbf{a}} = \begin{pmatrix} 1 & \mathbf{a} \\ 0 & id_F \end{pmatrix} \quad \text{and} \quad \Gamma_{\mathbf{b}} = \begin{pmatrix} 1 & 0 \\ \mathbf{b}^t & id_F \end{pmatrix}, \quad \text{where } F = A^r, \mathbf{a} \in JF, \mathbf{b} \in F.$$

Further, we will write $E^1(A \oplus M)$ for $E^1(A \oplus M, A)$.

Let $p \in M$ and $\varphi \in M^*$ be such that $\varphi(p) = 0$. Let $\varphi_p \in \text{End}(M)$ be defined as $\varphi_p(q) = \varphi(q)p$. Then $1 + \varphi_p$ is a (unipotent) automorphism of M . An automorphism of M of the form $1 + \varphi_p$ is called a *transvection* of M if either $p \in \text{Um}(M)$ or $\varphi \in \text{Um}(M^*)$. We denote by $E(M)$, the subgroup of $\text{Aut}(M)$ generated by all the transvections of M .

We begin with the following result due to Bak-Basu-Rao ([3], theorem 3.10). In [5], we have proved results for $E^1(A \oplus P)$. Due to this result, we can interchange $E(A \oplus P)$ and $E^1(A \oplus P)$.

Theorem 2.1 *Let A be a ring and P a projective A -module of rank ≥ 2 . Then $E^1(A \oplus P) = E(A \oplus P)$.*

We state two result ([8], Corollaries 2.7) and ([7], Theorem 2.5) due to Heitmann and Ellouz-Lombardi-Yengui respectively. The first one generalizes Bass's cancellation [4] to non-Noetherian case and the second one generalizes Quillen-Suslin theorem [15, 19] to all 0-dimensional rings.

Theorem 2.2 *Let A be a ring of dimension d and P a projective A -module of rank $\geq d + 1$. Then $E(A \oplus P)$ acts transitively on $\text{Um}(A \oplus P)$. In particular, P is cancellative.*

Theorem 2.3 *Let R be a ring of dimension 0. Then all projective modules over $R[X_1, \dots, X_n]$ are free.*

We will state five results. They are stated and proved with the assumption that rings are Noetherian. But the same proof works for non-Noetherian rings.

Lemma 2.4 ([6], Remark 2.2) *Let A be a ring, I an ideal of A and P a projective A -module. Then the natural map $E(A \oplus P) \rightarrow E(\frac{A \oplus P}{I(A \oplus P)})$ is surjective.*

Lemma 2.5 ([5], Lemma 3.1) *Let A be a ring and P a projective A -module. Let “bar” denote reduction modulo the nil-radical of A . For an ideal J of A , if $E^1(\overline{A \oplus P}, \overline{J})$ acts transitively on $\text{Um}^1(\overline{A \oplus P}, \overline{J})$, then $E^1(A \oplus P, J)$ acts transitively on $\text{Um}^1(A \oplus P, J)$.*

Lemma 2.6 ([12], Lemma 1.1) *Let A be a reduced ring and P an A -module. Assume $s \in A$ is a non-zerodivisor such that P_s is free of rank $r \geq 1$. Then there exist $p_1, \dots, p_r \in P$, $\phi_1, \dots, \phi_r \in P^*$ and $t \in \mathbb{N}$ such that*

- (i) $s^t P \subset F$ and $s^t P^* \subset G$ with $F = \sum_1^r A p_i$ and $G = \sum_1^r A \phi_i$.
- (ii) $(\phi_i(p_j))_{1 \leq i, j \leq r} = \text{diagonal}(s^t, \dots, s^t)$.

Lemma 2.7 ([5] Lemma 3.10) *Let A be a reduced ring and P a projective A -module of rank r . Assume there exist a non-zerodivisor $s \in A$ such that P_s is free. Choose $p_1, \dots, p_r \in P$, $\varphi_1, \dots, \varphi_r \in P^*$ satisfying (2.6). Let $(a, p) \in \text{Um}(A \oplus P, sA)$ with $p = c_1 p_1 + \dots + c_r p_r$, where $c_i \in sA$ for all i . Assume there exist $\phi \in E_{r+1}^1(A, sA)$ such that $\phi(a, c_1, \dots, c_r) = (1, 0, \dots, 0)$. Then there exist $\Phi \in E(A \oplus P)$ such that $\Phi(a, p) = (1, 0)$.*

Lemma 2.8 ([21], Lemma 4.2) *Let A be a reduced ring and P an A -module. Assume there exist non-zerodivisors $s_1, \dots, s_r \in A$, $p_1, \dots, p_r \in P$ and $\phi_1, \dots, \phi_r \in P^*$ such that $(\phi_i(p_j))_{r \times r} = \text{diagonal}(s_1, \dots, s_r) := N$. Let \mathcal{M} be the subgroup of $\text{GL}_r(A)$ consisting of all matrices $I + TN^2$ for $T \in M_r(A)$. Then the map*

$$\Phi : \mathcal{M} \rightarrow \text{Aut}_A(P); \quad \Phi(I + TN^2) = \text{id}_P + (p_1, \dots, p_r) T N (\phi_1, \dots, \phi_r)^t$$

is a group homomorphism.

The following result is from Lam’s book ([10], Proposition VI.1.14).

Proposition 2.9 *Let B be a ring and $a, b \in B$ two comaximal elements. Then for any $\sigma \in E_n(B_{ab})$ with $n \geq 3$, there exist $\alpha \in E_n(B_b)$ and $\beta \in E_n(B_a)$ such that $\sigma = (\alpha)_a(\beta)_b$.*

We state two results due to Yengui [22] and Abedelfatah [1] respectively.

Theorem 2.10 *Let A be a ring of dimension d and $n \geq d + 2$. Then*

- (i) $E_n(A[X])$ acts transitively on $\text{Um}_n(A[X])$.
- (ii) $E_n(A[X, X^{-1}])$ acts transitively on $\text{Um}_n(A[X, X^{-1}])$.

We recall the following well known result.

Lemma 2.11 *Let R be a reduced ring. Then $\dim R = 0$ if and only if $R_{\mathfrak{m}}$ is a field for every maximal ideal \mathfrak{m} of R .*

Assume R is a reduced Noetherian ring having no non-zerodivisor. Then it is easy to see that R is a finite direct product of fields. In particular, $\dim R = 0$. The same result is true for non-Noetherian ring. We thank I. Yengui for bringing this result to our notice. As we are unable to find a reference, we will give a proof for completeness.

Lemma 2.12 *Let R be a reduced rings and S the set of all non-zerodivisors of R . Then $\dim S^{-1}R = 0$.*

Proof Write $A = S^{-1}R$. Then A is a reduced ring having no non-zero-divisor. To show $\dim A = 0$, by (2.11), it is enough to show that $A_{\mathfrak{m}}$ is a field for every maximal ideal \mathfrak{m} of A . Hence, we may assume that (A, \mathfrak{m}) is a local reduced ring having no non-zero-divisor. Assume contrary that there exist $x \in \mathfrak{m} - 0$.

Let $\Sigma_1 = \{\mathfrak{p} \in \text{Spec}(A) \mid x \in \mathfrak{p}\}$ and $\Sigma_2 = \text{Spec}(A) - \Sigma_1$. Since A is reduced, $\Sigma_2 \neq \emptyset$ (otherwise x is nilpotent). Let I and J be the intersections of all prime ideals in Σ_1 and Σ_2 respectively. Since A is reduced, $I \cap J = (0)$. In particular, $IJ = (0)$.

As $x \notin \mathfrak{p}$ for every $\mathfrak{p} \in \Sigma_2$, we get $(0 : x) \subset J$. Since A_x is also a reduced ring, we get $\text{nilrad}(A_x) = JA_x = (0)$. Hence, if $z \in J$, then $x^r z = 0$ for some r . This gives $xz = 0$ (A is reduced). Therefore, we get $J \subset (0 : x)$ and hence $J = (0 : x)$.

Since x is a zero-divisor, we get $(0 : x) = J \neq 0$. Choose a non-zero element $y \in J$. Then $xy = 0$. Note that $x - y \neq 0$ (otherwise $x^2 = 0$) and $x - y \in \mathfrak{m}$. Hence $x - y$ is also a zero-divisor. If $u \in \mathfrak{m}$ is such that $u(x - y) = 0$, then $ux(x - y) = 0$ gives $ux^2 = 0$. This gives $ux = 0$ (A is reduced) and hence $uy = 0$. Therefore, we get $(0 : x - y) \subset (0 : x) \cap (0 : y)$.

As above, we can see that $(0 : y)$ is the intersection of all prime ideals \mathfrak{p} such that $y \notin \mathfrak{p}$. Hence $(0 : x) \cap (0 : y)$ is the intersection of all prime ideals \mathfrak{p} such that $x, y \notin \mathfrak{p}$. But $xy = 0$, hence every prime ideal \mathfrak{p} contains either x or y . Hence $(0 : x) \cap (0 : y) = A$. In particular, $x = 0$, a contradiction. \square

3 Zero dimension case

Proposition 3.1 *Let $\Sigma(n)$ be a set of rings which is closed w.r.t. following properties:*

(i) *If $R \in \Sigma(n)$ and $f \in R[Y]$, then $R[Y]_{f(1+fR[Y])} \in \Sigma(n)$.*

(ii) *If $R \in \Sigma(n)$, then all projective modules over $R[Y_1, \dots, Y_n]$ are free.*

Then, for $R \in \Sigma(n)$, all projective modules over $R[Y_1, \dots, Y_n, (f_1 \cdots f_m)^{-1}]$ are free, where $m \leq n$ and $f_i \in R[Y_i]$.

Proof Let P be a projective $A = R[Y_1, \dots, Y_n, (f_1 \cdots f_m)^{-1}]$ -module of rank r . If $m = 0$, then P is free by assumption (ii). Assume $m > 0$ and use induction on m . Write $C = R[Y_1, \dots, Y_n, (f_1 \cdots f_{m-1})^{-1}]$ and $S = 1 + f_m R[Y_m]$. Then $A = C_{f_m}$, $B = R[Y_m]_{f_m S} \in \Sigma$ and $S^{-1}A = B[Y_1, \dots, Y_{m-1}, Y_{m+1}, \dots, Y_n, (f_1 \cdots f_{m-1})^{-1}]$. By induction on m , $S^{-1}P$ is free. Since P is finitely generated, we can find $g \in S$ such that P_g is free. Note that f_m and g are comaximal elements of $R[Y_m]$. Consider the fiber product diagram

$$\begin{array}{ccc} C & \longrightarrow & C_{f_m} = A \\ \downarrow & & \downarrow \\ C_g & \longrightarrow & C_{f_m g} = A_g \end{array}$$

Patching projective modules P over C_{f_m} and $(C_g)^r$ over C_g , we get that $P \xrightarrow{\sim} Q_{f_m}$, where Q is a projective C -module of rank r . By induction on m , projective modules over C are free. Hence Q is free and therefore P is free. \square

Let $\Sigma(n)$ be the set of all rings of dimension 0. If $R \in \Sigma(n)$ and $f \in R[Y]$, then $\dim R[Y] = 1$ and $\dim R[Y]_{f(1+fR[Y])} = 0$. Hence $R[Y]_{f(1+fR[Y])} \in \Sigma(n)$. Using (2.3), projective modules over $R[Y_1, \dots, Y_n]$

are free. Hence $\Sigma(n)$ satisfies the hypothesis (i, ii) of (3.1). Therefore, we get the following result which generalizes (2.3).

Proposition 3.2 *Let R be a ring of dimension 0 and $A = R[Y_1, \dots, Y_n, (f_1 \cdots f_m)^{-1}]$, where $m \leq n$ and $f_i \in R[Y_i]$. Then all projective A -modules are free.*

The following result is due to Abedelfatah [2] in polynomial ring ($m = 0$) case. Note that if $E_r(R[Y_1, \dots, Y_n])$ acts transitively on $\text{Um}_r(R[Y_1, \dots, Y_n])$, then for every $m \leq n$, $E_r(R[Y_1, \dots, Y_m])$ acts transitively on $\text{Um}_r(R[Y_1, \dots, Y_m])$.

Theorem 3.3 *Let R be a ring of dimension 0 and $A = R[Y_1, \dots, Y_n, (f_1 \cdots f_m)^{-1}]$, where $m \leq n$ and $f_i \in R[Y_i]$. Then $E_r(A)$ acts transitively on $\text{Um}_r(A)$ for $r \geq 3$.*

Proof Given $v \in \text{Um}_r(A)$, we have to find $\Phi \in E_r(A)$ such that $\Phi(v) = e_1 = (1, 0, \dots, 0)$. The case $m = 0$ is due to Abedelfatah [2]. Assume $m > 0$ and use induction on m . If $C = R[Y_1, \dots, Y_n, (f_1 \cdots f_{m-1})^{-1}]$, then $A = C_{f_m}$. Let $S = 1 + f_m R[Y_m]$. Then $S^{-1}A = B[Y_1, \dots, Y_{m-1}, Y_{m+1}, \dots, Y_n, (f_1 \cdots f_{m-1})^{-1}]$, where $B = R[Y_m]_{f_m S}$ is 0 dimensional. By induction on m , $E_r(S^{-1}A)$ acts transitively on $\text{Um}_r(S^{-1}A)$. Hence, there exist $\sigma \in E_r(S^{-1}A)$ such that $\sigma(v) = e_1$. We can find $g \in S$ and $\tilde{\sigma} \in E_r(C_{f_m g})$ such that $\tilde{\sigma}(v) = e_1$. Note that f_m and g are comaximal elements of $R[Y_m]$. Consider the fiber product diagram

$$\begin{array}{ccc} C & \longrightarrow & C_{f_m} = A \\ \downarrow & & \downarrow \\ C_g & \longrightarrow & C_{f_m g} = A_g \end{array}$$

By (2.9), $\tilde{\sigma}$ has a splitting $\tilde{\sigma} = (\alpha)_{f_m}(\beta)_g$, where $\alpha \in E_r(C_g)$ and $\beta \in E_r(C_{f_m})$. We have unimodular elements $\beta(v) \in \text{Um}_r(C_{f_m})$ and $\alpha^{-1}(e_1) \in \text{Um}_r(C_g)$ whose images in $C_{f_m g}$ are same. Hence, patching $\beta(v)$ and $\alpha^{-1}(e_1)$, we get $w \in \text{Um}_r(C)$ such that its image in C_{f_m} is $\beta(v)$. By induction on m , $E_r(C)$ acts transitively on $\text{Um}_r(C)$. Hence, we can find $\phi \in E_r(C)$ such that $\phi(w) = e_1$. If $\Phi_1 \in E_r(C_{f_m})$ is the image of ϕ , then $\Phi_1(\alpha(v)) = e_1$. Write $\Phi = \Phi_1 \alpha \in E_r(A)$, we are done. \square

4 Main Theorem

The following result is proved in ([9], Lemma 3.3) with the assumption that ring is Noetherian. Using (2.8) and following the same proof, we get the same result for non-Noetherian ring.

Lemma 4.1 *Let A be a reduced ring and P a projective A -module of rank r . By (2.12), choose a non-zerodivisor $s \in A$ such that (2.6) holds. Assume that R^r is cancellative, where $R = A[X]/(X^2 - s^2 X)$. Then $\text{Aut}(A \oplus P, sA)$ acts transitively on $\text{Um}^1(A \oplus P, s^2 A)$.*

An immediate consequence of (4.1) is the following result. Its proof is same as of ([9], Corollary 3.5) using (2.2). This result says the following: Assume Σ is a set of rings of dimension d which is closed w.r.t. finite integral extension. If R^d is cancellative for all $R \in \Sigma$, then for all $R \in \Sigma$, projective R -modules of rank d are also cancellative.

Corollary 4.2 *Let A be a reduced ring of dimension d and P a projective A -module of rank d . Using (2.12), choose a non-zerodivisor $s \in A$ such that (2.6) holds. Assume B^d is cancellative, where $B = A[X]/(X^2 - s^2X)$. Then P is cancellative.*

Let R be a ring and I an ideal of R . For $n \geq 3$, let $E_n(I)$ be the subgroup of $E_n(R)$ generated by $E_{ij}(a) = I + ae_{ij}$ with $a \in I$, $1 \leq i \neq j \leq n$ and only non-zero entry of the matrix ae_{ij} is a at the (i, j) th place. Let $E_n(R, I)$ denote the normal closure of $E_n(I)$ in $E_n(R)$. Then we have two characterisation of $E_n(R, I)$ due to Suslin-Vaserstein [20] and Stein [18] respectively.

Proposition 4.3 *The kernel of the natural map $E_n(R) \rightarrow E_n(R/I)$ is isomorphic to $E_n(R, I)$.*

Proposition 4.4 *Consider the following fiber product diagram*

$$\begin{array}{ccc} R(I) & \xrightarrow{p_1} & R \\ p_2 \downarrow & & \downarrow j_1 \\ R & \xrightarrow{j_2} & R/I \end{array}$$

Then $E_n(R, I)$ is the kernel of the natural surjection $E_n(p_1) : E_n(R(I)) \twoheadrightarrow E_n(R)$.

The following result is proved in ([6], Lemma 3.3) with the assumption that ring is Noetherian. Using (4.3, 4.4, 2.7) and following the proof of ([6], Lemma 3.3), we get the same result for non-Noetherian ring.

Lemma 4.5 *Let A be a reduced ring and P a projective A -module of rank r . Using (2.12), choose a non-zerodivisor $s \in A$ such that (2.6) holds. Assume $E_{r+1}(B)$ acts transitively on $\text{Um}_{r+1}(B)$, where $B = A[X]/(X^2 - s^2X)$. Then $E(A \oplus P)$ acts transitively on $\text{Um}(A \oplus P, s^2A)$.*

The proof of the following result is same as of ([6], Theorem 3.4) using (4.5, 2.2). Note that (4.6 and 4.7) holds for arbitrary rings, since using (2.5), we can assume that ring is reduced.

Proposition 4.6 *Let $A \in \mathcal{R}$ be a reduced ring of dimension d and P a projective A -module of rank $r \geq d$. Using (2.12), choose a non-zerodivisor $s \in A$ such that (2.6) holds. Assume $E_{r+1}(B)$ acts transitively on $\text{Um}_{r+1}(B)$, where $B = A[X]/(X^2 - s^2X)$. Then $E(A \oplus P)$ acts transitively on $\text{Um}(A \oplus P)$.*

Lemma 4.7 *Let R be a reduced ring of dimension d and P a projective $A = R[Y_1, \dots, Y_n, (f_1 \cdots f_m)^{-1}]$ -module of rank $\geq d + 1$, where $m \leq n$ and $f_i \in R[Y_i]$. Using (2.12, 3.2), choose a non-zerodivisor $s \in R$ such that (2.6) holds. Assume $E_{r+1}(B)$ acts transitively on $\text{Um}_{r+1}(B)$, where $B = A[X]/(X^2 - s^2X)$. Then $E(A \oplus P)$ acts transitively on $\text{Um}(A \oplus P)$.*

Proof First assume that $d = 0$. Then P is free, by (3.2), and we are done by (3.3). Assume $d > 0$ and use induction on d . Let $(a, p) \in \text{Um}(A \oplus P)$ and “bar” denote reduction modulo s^2A . By induction on d , $E(\overline{A \oplus P})$ acts transitively on $\text{Um}(\overline{A \oplus P})$ (Note that by (2.5), we may assume that \overline{A} is reduced). Hence, we can find $\overline{\sigma} \in E(\overline{A \oplus P})$ such that $\overline{\sigma}(\overline{a}, \overline{p}) = (1, 0)$. By (2.4), lifting $\overline{\sigma}$ to an element $\theta \in E(A \oplus P)$, we get

$\theta(a, p) \in \text{Um}(A \oplus P, s^2 A)$. Apply (4.5), we get $\tilde{\theta} \in E(A \oplus P)$ such that $\tilde{\theta}\theta(a, p) = (1, 0)$. This completes the proof. \square

As a consequence of (4.7), we will prove the main result of this section which generalizes Yengui and Abedelfatah (2.10). If $d = 0$, then P is free (by 3.2) and we can use (2.10). Hence, we may assume $d \geq 1$.

Theorem 4.8 *Let R be a ring of dimension d and A is one of $R[Y]$ or $R[Y, Y^{-1}]$. Let P be a projective A -module of rank $r \geq d + 1$. Then $E(A \oplus P)$ acts transitively on $\text{Um}(A \oplus P)$.*

Proof Using (2.5), we may assume that R is reduced. If S is the set of all non-zerodivisors of R , then $\dim S^{-1}R = 0$. Hence $S^{-1}P$ is free, by (3.2). Therefore, we can choose a non-zerodivisor $s \in R$ such that (2.6) holds. If $R' = R[X]/(X^2 - s^2 X)$, then $\dim R' = d$. If $B = A[X]/(X^2 - s^2 X)$, then B is one of $R'[Y]$ or $R'[Y, Y^{-1}]$. By (2.10), $E_{r+1}(B)$ acts transitively on $\text{Um}_{r+1}(B)$. Hence, using (4.7), we get $E(A \oplus P)$ acts transitively on $\text{Um}(A \oplus P)$. \square

By a result of Seidenberg ([17], Theorem 4), if R is a Prufer domain, then $\dim R[Y_1, \dots, Y_n] = \dim R + n$. Hence (4.9) holds for a Prufer domain R and generalizes (4.8).

Proposition 4.9 *Let R be a ring of dimension d such that $\dim R[Y] = d + 1$. Let $A = R[Y, f^{-1}]$ with $f \in R[Y]$ and P a projective A -module of rank $r \geq d + 1$. Then $E(A \oplus P)$ acts transitively on $\text{Um}(A \oplus P)$.*

Proof Using (2.5), we may assume that R is reduced. Note that if $d = 0$, then P is free, by (3.2). In this case, we are done, by (3.3). Hence, we may assume that $d \geq 1$. Using (2.12), choose a non-zerodivisor $s \in R$ such that (2.6) holds. Write $R' = R[X]/(X^2 - s^2 X)$ and $B = R'[Y, f^{-1}]$. Using (4.7), it is enough to show that $E_{r+1}(B)$ acts transitively on $\text{Um}_{r+1}(B)$.

Let $v \in \text{Um}_{r+1}(B)$. If we write $C = R'[Y]$, then $B = C_f$. Since R' is an integral extension of R , $\dim R[Y] = d + 1 = \dim R'[Y]$. Hence $\dim C_{f(1+fR'[Y])} = d$. Applying (2.2), we get $\sigma \in E_{r+1}(C_{f(1+fR'[Y])})$ such that $\sigma(v) = (1, \dots, 0)$. We can find $g \in 1 + fR'[Y]$ and $\tilde{\sigma} \in E_{r+1}(C_{fg})$ such that $\tilde{\sigma}(v) = (1, 0, \dots, 0)$.

By (2.9), $\tilde{\sigma}$ has a splitting $\tilde{\sigma} = (\alpha)_f(\beta)_g$, where $\alpha \in E_{r+1}(C_g)$ and $\beta \in E_{r+1}(C_f)$. We have two unimodular elements $\beta(v) \in \text{Um}_{r+1}(C_f)$ and $\alpha^{-1}((1, 0, \dots, 0)) \in \text{Um}_{r+1}(C_g)$ whose images in C_{fg} are same. Hence, patching $\beta(v)$ and $\alpha^{-1}((1, 0, \dots, 0))$, we get $w \in \text{Um}_{r+1}(C)$ whose image in C_f is $\beta(v)$. By Yengui (2.10), $E_{r+1}(C)$ acts transitively on $\text{Um}_{r+1}(C)$. Hence, we can find $\phi \in E_{r+1}(C)$ such that $\phi(w) = (1, 0, \dots, 0)$. If Φ_1 is the image of ϕ in C_f , then $\Phi_1(\alpha(v)) = (1, 0, \dots, 0)$ and $\Phi_1\alpha \in E_{r+1}(B)$. This proves the result. \square

Lequain-Simis have shown [11] that if R is a Prufer domain, then projective modules over $R[Y_1, \dots, Y_n]$ are extended from R . In particular, if R is a valuation domain, then projective $R[Y_1, \dots, Y_n]$ -modules are free. It is natural to ask if projective modules over $R[Y_1, \dots, Y_n, (f_1 \cdots f_m)^{-1}]$ are free, where R is a valuation domain, $m \leq n$ and $f_i \in R[Y_i]$. We give a partial answer to above question.

Proposition 4.10 *Let R be a valuation domain of dimension d and $A = R[X, Y, f^{-1}]$ with $f \in R[Y]$. Then stably free A -modules of rank $\geq d + 1$ are free.*

Proof Let P be a stably free A -module of rank $r \geq d + 1$. If $d = 0$, then P is free, by (3.2). Hence assume $d \geq 1$. If $C = R[X, Y]$ and $S = 1 + fR[Y]$, then $\dim R[Y] = d + 1$, by Seidenberg [17] and hence $\dim R[Y]_{fS} = d$. Further, $S^{-1}A = R[Y]_{fS}[X]$. Hence, by (2.10), $S^{-1}P$ is free. We can choose $g \in S$ such that P_g is free. In the fiber product diagram considered earlier, patching projective modules P over C_f and $(C_g)^r$ over C_g , we get a projective C -module Q such that $P \xrightarrow{\sim} Q_f$. Since every projective C -module is free, by [11], Q and hence P is free. \square

References

- [1] A. Abedelfatah, *On stably free modules over Laurent polynomial rings*, Proc. A.M.S. **139** (2011) 4199-4206.
- [2] A. Abedelfatah, *On the action of elementary group on the unimodular rows*, J. Algebra **368** (2012) 300-304.
- [3] A. Bak, R. Basu, Ravi A. Rao, *Local global principle for transvection groups*, Proc. Amer. Math. Soc. **138** (2010) 1191-1204.
- [4] H. Bass, *K-theory and stable algebra*, Publ. Math. Inst. Hautes Etudes Sci. **22** (1964) 5-60.
- [5] A.M. Dhorajia and M.K. Keshari, *Projective modules over overrings of polynomial rings*, J. Algebra **323** (2010) 551-559.
- [6] A.M. Dhorajia and M.K. Keshari, *A note on cancellation of projective modules* J.P.A.A. **216** (2012) 126-129.
- [7] A. Ellouz, H. Lombardi and I. Yengui, *A constructive comparison of rings $R(X)$ and $R\langle X \rangle$ and application to Lequain-Simis induction theorem*, J. Algebra **320** (2008) 521V533.
- [8] R. Heitmann, *Generating non-noetherian modules efficiently*, Michigan Math J. **31** (1984) 167-180.
- [9] M.K. Keshari, *Cancellation problem for projective modules over affine algebras*, J. K-Theory **3** (2009) 561-581.
- [10] T.Y. Lam, *Serre's problem on projective modules*, Springer Monograph in Mathematics, Springer-Verlag, Berlin, 2006.
- [11] Y. Lequain and A. Simis, *Projective modules over $R[X_1, \dots, X_n]$, R a Prufer domain*, J.P.A.A. **18** (1980) 165-171.
- [12] H. Lindel, *Unimodular elements in projective modules*, J. Algebra **172** (1995) 301-319.
- [13] S. Mandal, *Basic elements and cancellation over laurent polynomial rings*, J. Algebra **79** (1982) 251-257.
- [14] B. Plumstead, *The conjectures of Eisenbud and Evans*, Amer. J. Math. **105** (1983) 1417-1433.
- [15] D. Quillen, *Projective modules over polynomial rings*, Invent. Math. **36** (1976) 167-171.
- [16] Ravi A. Rao, *A question of H. Bass on the cancellative nature of large projective modules over polynomial rings*, Amer. J. Math. **110** (1988) 641-657.
- [17] A. Seidenberg, *On the dimension theory of rings II*, Pacific J. Math. **4** (1954) 603-614.
- [18] M.R. Stein, *Relativizing functors on rings and algebraic K-theory*, J. Algebra **19** (1971) 140-152.
- [19] A.A. Suslin, *Projective modules over a polynomial rings are free*, Sov. Math. Dokl. **17** (1976) 1160-1164.
- [20] A.A. Suslin and L.N. Vaserstein, *Serre's problem on projective modules over polynomial rings and algebraic K-theory*, Math. USSR Izvestija **10** (1976) 937-1001.
- [21] A. Weimers, *Cancellation properties of projective modules over Laurent polynomial rings* J. Algebra **156** (1993), 108-124.
- [22] I. Yengui, *Stably free modules over $R[x]$ of rank $> \dim R$ are free*, Mathematics of Computation **80** no. 274 (2011) 1093-1098.